Course ISE 435: Distributed Algorithms in Network Communication
Recitation 2 Exercise

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1 Exercise 1

An undirected graph is called *bipartite* if its nodes can be colored using two colors (red and black) in such a way that no two nodes of the same color are adjacent.

Show that an undirected graph is bipartite if and only if it contains no cycles of odd length.

1.1 Proof

Two directions to prove - if and only if.

1.2 If: Bipartite → no odd cycles

Let us start by assuming the graph G is bipartite and has a single cycle of odd length. Let's consider just the induced subgraph with just the nodes in the odd cycle - adding the other nodes will not help. Assume the cycle is of length n and that the nodes in the cycle are ordered $n_0, \ldots n_{n-1}$. Let's start by coloring n_0 red, so n_1 must be black, n_2 must be red, etc., following the pattern that even numbered nodes are red and odd numbered nodes are black. The last node n_{n-1} must be red (n is odd, so $n-1$ is even). However, n_0 is also red and since n_0 and n_{n-1} are neighbors, that is illegal. Therefore G must not be bipartite, a contradiction to the assumption. Therefore, it must be the case that since G is bipartite, it does not have a cycle of odd length.

1.3 Only If: no odd cycles → Bipartite

For this we must show that any graph without an odd cycle is bipartite.

Let's start with a simple proof - that all trees are bipartite.

Lemma Start at the root of the tree G, color it red. Color each child of the root black. Color each grandchild of the root red, and so on. Each node at an odd depth from the root is colored black. Each node at an even depth of the root is colored red. Since there are no cycles, there are no nodes of ambiguous depths, so the coloring scheme is sufficient and shows that G is bipartite.

Now, using the lemma above, let's consider the graph G with no odd cycles. Let's start by making a spanning tree for the graph T. Since it's a tree, we have from the lemma that T is bipartite and we have a red-black coloring scheme for it.

Now, we have a problem if the coloring scheme for T is not good enough for G. That means that we have two nodes n and m which are the same color in T but are neighbors in G. Since they are the same color in T, that means the path must be of an even length - alternating colors along the way. Now if they are neighbors in G that means that there is an odd length cycle in G, a contradiction to our assumption that G has no odd cycles. Therefore, G must be bipartite if it has no odd cycles.

2 Exercise 2

Prove that (1) rings of even size, (2) trees, and (3) hypercubes are bipartite.

2.1 Proof

Try at home.